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UCSD Spring 2008

Social Choice Problem Set --- Suggested Answers

Problems 1 and 2 deal with a Bergson-Samuelson social welfare function. For the economy consisting of the set H of households, a Bergson-Samuelson social welfare functin can be described as

$$W(u^1(x^1), u^2(x^2), ..., u^{\#H}(x^{\#H}))$$

where u^i is household i's utility function and x^i is N-dimensional consumption vector. We assume

Positive association : $W_i > 0$ (the marginal social welfare of each household's utility is strictly positive; the subscript indicates a partial derivative)

Strict monotonicity: $u^i_k > 0$ (positive marginal utility for each good)

and assume an interior solution (just to make taking derivatives easy).

Consider a pure exchange economy, so that the resource constraint is

(1) $\sum_{i \in H} x^i = r$ where x^i and r are both positive N-dimensional vectors.

1. Show that any maximum of W subject to (1) is a Pareto efficient allocation (there are at least two ways to do this: you can develop the first order conditions to maximize W and show that they characterize a Pareto efficient allocation or go straight to the definitions).

Suggested Answer : From the definition: Consider the value $W(u^1(x^1), u^2(x^2), ..., u^{\#H}(x^{\#H}))$. If there is any Pareto improvement possible, then that improvement will increase the value of W (by Positive Association). Hence an optimizing value for W, x^{oi}, all i, has no room for additional Pareto improvement; it is Pareto efficient.

By calculation: The first order condition for optimizing W is

$$\left(\frac{\partial W}{\partial u^i(x^{oi})}\right)\left(\frac{\partial u^i}{\partial x^{oi}{}_n}\right) = \left(\frac{\partial W}{\partial u^j(x^{oj})}\right)\left(\frac{\partial u^j}{\partial x^{oj}{}_n}\right) \text{ for all i in H,}$$

n =1, 2, ..., N. And for m distinct from n, equivalently,

$$\left(\frac{\partial W}{\partial u^i(x^{oi})}\right)\left(\frac{\partial u^i}{\partial x^{oi}{}_m}\right)=\left(\frac{\partial W}{\partial u^j(x^{oj})}\right)\left(\frac{\partial u^j}{\partial x^{oj}{}_m}\right).$$ Dividing these expressions into one another, we have

$$\left(\frac{\partial u^i}{\partial x^{oi}{}_n}\right)\Big/\left(\frac{\partial u^i}{\partial x^{oi}{}_m}\right)=\left(\frac{\partial u^j}{\partial x^{oj}{}_n}\right)\Big/\left(\frac{\partial u^j}{\partial x^{oj}{}_m}\right)$$

But this is the first order condition for Pareto efficiency (equating marginal rates of substitution).

2. Consider W with linear weights:

$W(u^1(x^1), u^2(x^2), ..., u^{\#H}(x^{\#H})) = \sum_{i\in H} a^i u^i(x^i)$

where $a^i > 0$, is a real number. Demonstrate the following result.

Proposition: Let $x^{o1}, x^{o2}, ..., x^{o\#H}$, be a Pareto efficient allocation subject to (1). Then there is a choice of a^i, $i \in H$, so that $x^{o1}, x^{o2}, ..., x^{o\#H}$, is a maximum of W subject to (1).

Does the proposition mean that a family of Bergson-Samuelson social welfare functions can determine as their maxima the whole range of Pareto efficient points?

Suggested Answer : Rearranging terms in the answer to question 1, we have

$$\left(\frac{\partial W}{\partial u^i(x^{oi})}\right)\Big/\left(\frac{\partial W}{\partial u^j(x^{oj})}\right)=\left(\frac{\partial u^j}{\partial x^{oj}{}_n}\right)\Big/\left(\frac{\partial u^i}{\partial x^{oi}{}_n}\right)=a^i/a^j$$. That is, starting from the Pareto efficient allocation x^{oi} , find a^i for all i, by fullfilling this equation. Then x^{oi} is a welfare maximizer for that choice of a^i.

Yes. Thus for every Pareto efficient allocation, there is a Bergson-Samuelson social welfare function so that x^{oi} is the optimizing allocation.

Questions 3, 4, 5, and 6 deal with the Arrow Possibility Theorem. Eliminating any one of the four conditions (non-dictatorship, independence, Pareto principle, unrestricted domain) allows us to formulate a counterexample. That is, for any three of the four conditions, there is an Arrow Social Welfare Function that can fulfill those three. Demonstrate this result by finding a suitable Arrow SWF for each of the four sets of three conditions.

3. Find an Arrow SWF fulfilling non-dictatorship, independence, Pareto principle, but not unrestricted domain.
Suggested Answer Restrict the domain to single-peaked preferences. Then use majority voting on pairwise alternatives. Black's theorem shows that this will produce transitive social choice. Fails Unrestricted, since it only applies to single peaked profiles. Fulills Pareto since unanimous preference results in winning vote, fufills Nondictatorship since all views count equally, no one gets his way all the time. Fulfills Independence, since pairwise choice depends only on pairwise preferences.

4. Find an Arrow SWF fulfilling non-dictatorship, independence, unrestricted domain, but not Pareto principle.
Suggested Answer Choose any $P^* \in \Pi$. Let that P* be the social ordering. It will be transitive but the social choice rule will not fulfill Pareto principle. This amounts to imposing a constant ordering. Fails Pareto since no one's preferences count, fulfills Unrestricted since it covers all alternatives, Independence since no third alternative enters pairwise decisions, fulfills Nondictatorship, since no one gets his way under all profiles.

5. Find an Arrow SWF fulfilling non-dictatorship, Pareto principle, unrestricted domain, but not independence of irrelevant alternatives.
Suggested Answer : Borda count. Weighted voting. Fulills Pareto since unanimous preference results in a higher aggregate score, Unrestricted since it covers all preference profiles, fufills Nondictatorship since no one gets his way under all profiles. Fails Independence, since pairwise choice depends on intermediate ranking.

6. Find an Arrow SWF fulfilling independence, Pareto principle, unrestricted domain, but not non-dictatorship.
Suggested Answer Choose any $i^* \in H$. Set $P \equiv P_{i^*}$. P is now transitive and fulfills unrestricted domain and Pareto. But i* is now the dictator. Fulills Pareto since at least one household, i* , gets his way, fulfills Unrestricted

since the decision rule covers all profiles, Independence since no third alternative enters pairwise decisions. Fails Nondictatorship, since i* and only i* gets his way. .